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Atypical representations for type-I Lie superalgebras

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Abstract. A new modified induced module construction is presented for all finite-dimensional irreducible typical and atypical modules of a type-I basic classical Lie superalgebra. The method is illustrated with some low-dimensional representations of $gl(m|n)$ and all representations of $gl(2|1)$.

1. Introduction

The theory of Lie superalgebras and their representations plays a fundamental role in the understanding and exploitation of supersymmetry in physical systems. The concept of supersymmetry first arose in elementary particle physics (Wess and Zumino 1974) and has since been discussed in a variety of other areas including nuclear physics (Iachello 1980) and condensed matter physics (Parisi and Sourlas 1979, Nambu 1985). A comprehensive review of Lie superalgebra representation theory and its various physical applications is provided in Kostelecky and Campbell (1985).

Much of the necessary formal mathematical machinery for investigating basic classical Lie superalgebras and their finite-dimensional representations has now been developed, primarily by Kac (1977, 1978) who has introduced the now familiar categorisation of the irreducible representations into typical and atypical types. Typical representations have many properties in common with irreducible representations of simple Lie algebras. In particular, they are uniquely characterised by their infinitesimal characters and may be explicitly constructed by an induced module construction (Kac 1978) which leads directly to a simple determination of their dimensions and characters (Kac 1978). By contrast, the situation with atypical representations is far more complex. Various techniques have recently been introduced to gain greater insight into the structure of finite-dimensional irreducible atypical modules. We mention in particular the supertableaux methods (Balantekin and Bars 1981, 1982, King 1983, Hurni 1987, Dondi and Jarvis 1981, Farmer and Jarvis 1984) and those based on shift operators and weight space techniques (Hughes 1981, Van der Jeugt 1984, 1987, Hurni and Morel 1982, 1983). For a recent discussion on the calculation of characters of atypical modules see also Hughes and King (1987).

Despite this recent progress in the understanding of atypical representations, they are still far from well understood. One of the main difficulties is that a canonical construction for the irreducible atypical modules, analogous to Kac's construction for the typical modules, has not previously been developed. It is our aim in this paper to introduce a new direct method for the explicit construction of all atypical modules for a type-I basic classical Lie superalgebra. The method is based on a modification of

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the induced module construction of Kac (1978), in which our modules essentially appear as the unique irreducible submodules of lowest-weight Kac modules. The proposed construction contains a great deal of information on the structure of atypical modules and, in particular, may be applied to determine the $A(m|n)\downarrow A(m|n-1)$ (and hence $\mathrm{gl}(m|n)\downarrow \mathrm{gl}(m|n-1)$) branching rules (and thus characters) for atypical irreducible representations, as will be demonstrated for the case $\mathrm{gl}(n|l)$ in a forthcoming publication. It would be of interest to extend the construction of this paper to type-II basic classical Lie superalgebras.

2. Construction of irreducible modules

Let $L = L_0 \oplus L_1$ be a basic classical Lie superalgebra and let H be a fixed Cartan subalgebra (CSA) of the even part L_0, herein referred to as the CSA of L. We let $\Phi = \Phi_0 \cup \Phi_1$ denote the set of roots of L relative to H with Φ_0 (Φ_1) the set of even (odd) roots. For $\alpha \in \Phi$ we let $L_\alpha \subset L$ denote the corresponding root space of L and we denote by W_0 the Weyl group of L_0, herein referred to as the Weyl group of L.

Let B_0 be the Borel subalgebra of L_0 generated by the CSA H and positive root spaces $L_\alpha \subseteq L_0 (\alpha > 0)$ and let

$$B = B_0 \oplus B_1 \qquad (1)$$

be a fixed Borel subalgebra of L. We obtain the following decomposition of L:

$$L = N^- \oplus H \oplus N^+ \qquad B = H \oplus N^+ \qquad (2)$$

where $N^\pm$ are nilpotent subalgebras of L and $[H, N^\pm] \subseteq N^\pm$. A root $\alpha \in \Phi$ is called positive (negative) if $L_\alpha \subseteq N^+$ (N^-). We denote the set of positive roots of L by Φ^+ and we let Φ_0^+ (Φ_1^+) denote the subset of even (odd) positive roots: we have $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$, $\Phi = \Phi^+ \cup \Phi^-$, $\Phi^- = -\Phi^+$. Finally we let ρ_0 (ρ_1) denote the half-sum of the even (odd) positive roots and we set

$$\rho = \rho_0 - \rho_1$$

herein referred to as the graded half-sum of positive roots. Throughout we follow Kac (1978) and assume that the Borel subalgebra (1) is 'distinguished'.

We are concerned in this paper with simple basic classical Lie superalgebras of type I: that is, we assume L is one of the Lie superalgebras $A(m, n)$ or $C(n)$ (all other basic classical Lie superalgebras are said to be of type II). In such a case the Lie algebra L_0 is reductive and we have a decomposition

$$L_0 = [L_0, L_0] \oplus C$$

where $[L_0, L_0]$ is a semisimple Lie algebra and C is the one-dimensional centre of L_0 (except for $A(n, n)$ when $C = (0)$). The Lie superalgebra L also admits a natural Z-gradation (Kac 1978)

$$L = L_- \oplus L_0 \oplus L_+ \qquad L_1 = L_- \oplus L_+ \qquad (3)$$

where L_+ (L_-) is the subspace of L_1 spanned by the root spaces L_α corresponding to odd roots $\alpha \in \Phi_1^+$ (Φ_1^-). We note that the spaces $L_\pm$ constitute Abelian subalgebras of L:

$$[L_+, L_+] = [L_-, L_-] = (0).$$

In the notation of (1), we have $B_1 = L_+$ which constitutes the odd part of the distinguished Borel subalgebra B. In an analogous way the subalgebra $N = N^- \oplus H$ (cf (2)) also constitutes a Borel subalgebra of L whose corresponding odd part is given by L_-.

For each root $\alpha \in \Phi_1$ we choose a non-zero element x_α of the root space L_α: we note that x_α spans L_α. Following Kac (1978) we set

$$T_+ = \prod_{\alpha \in \Phi_1^+} x_\alpha \qquad T_- = \prod_{\alpha \in \Phi_1^+} x_{-\alpha} \qquad \text{(enveloping algebra product).} \tag{4}$$

We note that, since $L_\pm$ are Abelian algebras, the products in (4) are uniquely defined up to multiplication by ± 1. We note also that T_+ transforms, under Ad_{L_0}, as one-dimensional representation of L_0, and in particular must commute with the elements of the semisimple part $[L_0, L_0]$ of L_0. The operator T_- transforms, under Ad_{L_0}, contragrediently to T_+ which implies that the operators T_+T_-, T_-T_+ must commute with the elements of L_0.

Throughout we let U (U_0, $U_\pm$) denote the universal enveloping algebra of L (L_0, $L_\pm$). In view of (3) and the PBW theorem (Kac 1978) we have the following decomposition of U:

$$U = U_- U_0 U_+.$$

We note that the operators $T_\pm$ of (4) belong to $U_\pm$ respectively. Let us write

$$\Phi_1^+ = \{\alpha_1, \ldots, \alpha_k\} \qquad k = \tfrac{1}{2} \dim L_1.$$

Then the algebra U_+ is 2^k dimensional and is spanned by $I \in \mathbb{C}$ together with the basis monomials

$$x_{\alpha_{i_1}} x_{\alpha_{i_2}} \ldots x_{\alpha_{i_r}} \qquad 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k,\ 1 \leqslant r \leqslant k.$$

In a similar way U_- is spanned by $1 \in \mathbb{C}$ together with the basis monomials

$$x_{-\alpha_{i_1}} x_{-\alpha_{i_2}} \ldots x_{-\alpha_{i_r}} \qquad 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant k,\ 1 \leqslant r \leqslant k.$$

In the following we also find it convenient to introduce the subalgebras

$$\bar{L}_\pm = L_0 \oplus L_\pm \qquad L = L_- \oplus \bar{L}_+ = \bar{L}_- \oplus L_+.$$

We denote the universal enveloping algebras of $\bar{L}_\pm$ by $\bar{U}_\pm$ respectively.

As in the Lie algebra case, the finite-dimensional irreducible L-modules are uniquely characterised by their highest weights: we denote the irreducible L-module with highest weight Λ by $V(\Lambda)$, where Λ is necessarily a dominant integral weight (Humphreys 1972) of L_0. We denote the set of dominant integral weights of L_0 (and hence L) by D^+. Corresponding to any $\Lambda \in D^+$ we may construct a finite-dimensional indecomposable L-module with highest weight Λ, using the induced module construction of Kac (1978), as follows: let $V_0(\Lambda)$ denote the finite-dimensional irreducible L_0-module with highest weight Λ. We turn $V_0(\Lambda)$ into a $\bar{U}_+$-module by defining

$$L_+ V_0(\Lambda) = (0). \tag{5}$$

The induced L-module $\bar{V}(\Lambda)$ is then given by (Kac 1978)

$$\bar{V}(\Lambda) = U_- \oplus_{\bar{U}_+} V_0(\Lambda) = \bigoplus_{I \leqslant i_1 < i_2 < \ldots < i_r \leqslant k} x_{-\alpha_{i_1}} \ldots x_{-\alpha_{i_r}} \otimes V_0(\Lambda) \tag{6}$$

which constitutes an indecomposable L-module with highest weight Λ and dimension $\dim \bar{V}(\Lambda) = 2^k \dim V_0(\Lambda)$.

In a similar way we may define

$$L_- V_0(\Lambda) = (0) \tag{7}$$

which leads to the induced module

$$\bar{V}_{(-)}(\Lambda) = U_+ \otimes_{\bar{U}_-} V_0(\Lambda) \tag{8}$$

which is also indecomposable but in this case is cyclically generated by a lowest-weight vector of weight Λ_- where Λ_- is the lowest weight of $V_0(\Lambda)$: recall (Humphreys 1972) that $-\Lambda_-$ is the highest weight of the dual module $V_0^*(\Lambda)$ and Λ_- is W_0-conjugate to Λ.

If the induced module (6) is irreducible we necessarily have $\bar{V}(\Lambda) = V(\Lambda)$ in which case Λ is said to be *typical.* The structure of typical modules thus follows immediately from the induced module construction (6) which affords a great deal of information, and in particular enables a straightforward derivation of the dimensions and characters of typical L-modules (Kac 1978). However, in the case of *atypical* $\Lambda \in D^+$, the Kac module (6) is no longer irreducible and it is necessary to factor out by the (unique) maximal submodule of $\bar{V}(\Lambda)$, herein denoted by $M(\Lambda)$:

$$V(\Lambda) \simeq \bar{V}(\Lambda)/M(\Lambda).$$

In such a case the structure of $V(\Lambda)$ is difficult to determine from the induced module, since it is first necessary to construct the maximal submodule $M(\Lambda)$.

It is our aim here to develop an alternative direct construction of all finite-dimensional irreducible L-modules $V(\Lambda)$, using a modified induced module construction, which does not require a knowledge of the maximal submodule $M(\Lambda)$. Throughout, unless otherwise stated, $V_0(\Lambda)$ denotes a finite-dimensional irreducible L_0-module with highest weight Λ satisfying (5).

We note first that we may write

$$T_+T_- = \Delta + \Phi \qquad \Phi \in UL_+ \tag{9}$$

where

$$\Delta = [x_{\alpha_k}, [x_{\alpha_{k-1}}, \ldots [x_{\alpha_1}, T_-]\ldots]]. \tag{10}$$

We note that Δ necessarily belongs to the enveloping algebra U_0 of L_0 and moreover must commute with the elements of L_0. Thus Δ belongs to the centre Z_0 of U_0. It follows that T_+T_- must reduce to a scalar multiple of the identity, on the subspace $V_0(\Lambda) \subseteq \bar{V}(\Lambda)$, given by

$$T_+T_-\omega = \chi_\Lambda(\Delta)\omega \qquad \omega \in V_0(\Lambda) \tag{11}$$

where $\chi_\Lambda(\Delta)$ denotes the eigenvalue of $\Delta \in Z_0$ on $V_0(\Lambda)$. In view of Harish-Chandra's theorem Δ determines a polynomial function f_Δ on H^* defined by

$$f_\Delta(\Lambda) = \chi_\Lambda(\Delta)$$

which is necessarily fixed by all elements of the translated Weyl group $\tilde{W}_0$ (Humphreys 1972), viz

$$f_\Delta(\sigma(\Lambda + \rho_0) - \rho_0) = f_\Delta(\Lambda) \qquad \forall \sigma \in W_0.$$

We note also that f_Δ is to determine a polynomial of degree

$$k = |\Phi_1^+|.$$

Following the argument of Kac (1978), let α_s denote the distinguished simple odd root of Φ^+, so that

$$[x_\alpha, x_{-\alpha_s}] = 0 \qquad \alpha \in \Phi_0^+ \tag{12}$$

and let v_+^Λ be the maximal weight vector of $V_0(\Lambda)$. It follows immediately from (12) that if $(\Lambda, \alpha_s) = 0$ then

$$v_0 = x_{-\alpha_s} v_+^\Lambda \tag{13}$$

satisfies

$$\mathrm{B} v_0 = 0$$

and thus is a maximal weight state of L. In such a case the vector (13) cyclically generates an indecomposable L-module of highest weight $\Lambda - \alpha_s$. On the other hand we note that

$$T_- v_+^\Lambda \in \mathrm{U} x_{-\alpha_s} v_+^\Lambda = \bar{\mathrm{U}}_- v_0$$

(since T_- contains a factor $x_{-\alpha_s}$) from which we obtain

$$T_+ T_- v_+^\Lambda \in \bar{\mathrm{U}}_- v_0 .$$

Equation (11) then implies that, when $(\Lambda, \alpha_s) = 0$,

$$T_+ T_- v_+^\Lambda \in \bar{\mathrm{U}}_- v_0 \cap V_0(\Lambda) = (0)$$

from which we deduce that the polynomial function f_Δ is divisible by a factor

$$(\Lambda, \alpha_s) = (\Lambda + \rho, \alpha_s)$$

where we have used the fact that $(\rho, \alpha_s) = 0$. Using the $\tilde{\mathrm{W}}_0$-invariance of f_Δ we then deduce divisibility of f_Δ by factors

$$(\Lambda + \rho, \alpha) \qquad \alpha \in \Phi_1^+$$

which follows from the W_0-invariance of Φ_1^+ (i.e. W_0 permutes the roots of Φ_1^+). The number of such factors equals precisely the degree k of f_Δ from which we obtain

$$\chi_\Lambda(\Delta) = f_\Delta(\Lambda) = c \prod_{\alpha \in \Phi_1^+} (\Lambda + \rho, \alpha) \tag{14}$$

for some non-zero scalar $c \in \mathbb{C}$, in agreement with the result of Kac.

Before proceeding to Kac's main result on typical modules we need the following technical lemma.

Lemma 1.

(i) Every L-submodule of $\bar{V}(\Lambda)$ contains the L_0-module $T_- \otimes V_0(\Lambda)$.

(ii) Every L-submodule of $\bar{V}_{(-)}(\Lambda)$ contains the L_0-module $T_+ \otimes V_0(\Lambda)$.

Proof. Let

$$\omega = \sum_{1 \leq i_1 < \ldots < i_r \leq k} x_{-\alpha_{i_1}} \ldots x_{-\alpha_{i_r}} \otimes v_{i_1 i_2 \ldots i_r} \qquad v_{i_1 i_2 \ldots i_r} \in V_0(\Lambda)$$

be an arbitrary element of $\bar{V}(\Lambda)$. Then choose index r minimal with respect to the property $v_{i_1 \ldots i_r} \neq 0$ for some choice of r indices $1 \leq i_1 < \ldots < i_r \leq k$. It is convenient to renumber the odd positive roots according to

$$\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}, \alpha_{i_{r+1}}, \ldots, \alpha_{i_k} .$$

Then, by our construction, we have

$$\begin{aligned} x_{-\alpha_{i_{r+1}}} \ldots x_{-\alpha_{i_k}} \omega &= x_{-\alpha_{i_{r+1}}} \ldots x_{-\alpha_{i_k}} x_{-\alpha_{i_1}} \ldots x_{-\alpha_{i_r}} \otimes v_{i_1 i_2 \ldots i_r} \\ &= \pm T_- \otimes v_{i_1 \ldots i_r} . \end{aligned}$$

It follows that the L-module generated by ω must contain a non-zero vector $T_-\otimes v$, $v\in V_0(\Lambda)$, and thus the entire L_0-module $T_-\otimes V_0(\Lambda)$. Since $\omega\in\bar{V}(\Lambda)$ was chosen arbitrarily, part (i) immediately follows. In a similar way we may prove part (ii).

We thus arrive at the following result due to Kac (1978).

Theorem 1 (*Kac*). The induced module $\bar{V}(\Lambda)$, $\Lambda\in D^+$, is irreducible if and only if $(\Lambda+\rho,\alpha)\neq 0$, $\forall\alpha\in\Phi_1^+$.

Proof. Following Kac, since every submodule of $\bar{V}(\Lambda)$ contains the subspace $T_-\otimes V_0(\Lambda)$ it follows that $\bar{V}(\Lambda)$ is irreducible if and only if $T_+(T_-\otimes V_0(\Lambda))\neq(0)$. On the other hand we have, for $v\in V_0(\Lambda)$,

$$T_+T_-\otimes v=\chi_\Lambda(\Delta)v=c\prod_{\alpha\in\Phi_1^-}(\Lambda+\rho,\alpha)v$$

from which the result follows.

We remark that the above theorem obviously extends to the minimal-weight induced modules $\bar{V}_{(-)}(\Lambda)$ of (8). It follows that an irreducible module $V(\Lambda)$ is typical if and only if $(\Lambda+\rho,\alpha)\neq 0$, $\forall\alpha\in\Phi_1^+$, which is the main criterion of typicality due to Kac (1978). In such a case, as mentioned previously, the structure of typical modules follows from the induced module construction of (6).

More importantly, from our point of view, is the result of lemma 1 which enables one to construct all irreducible atypical modules. This follows from the fact that, if $V_0(\Lambda)$ satisfies (5), then lemma 1 implies that the L-module generated by $T_-\otimes V_0(\Lambda)$ is necessarily irreducible (with lowest weight $\Lambda_- - 2\rho_1$). Similarly, if $V_0(\Lambda)$ satisfies (7) then $T_+\otimes V_0(\Lambda)$ generates an irreducible L-module with highest weight $\Lambda+2\rho_1$.

To construct an irreducible L-module with highest weight $\Lambda\in D^+$ we note, since $(\rho_1,\alpha)=0$ for $\alpha\in\Phi_0^+$, that $\Lambda-2\rho_1\in D^+$. Thus we introduce the finite-dimensional irreducible L_0-module $V_0(\Lambda-2\rho_1)$ which we convert to a $\bar{U}_-$ module via

$$L_-V_0(\Lambda-2\rho_1)=(0).$$

We then consider the (lowest-weight) induced module

$$\bar{V}_{(-)}(\Lambda-2\rho_1)=U_+\otimes_{\bar{U}_-}V_0(\Lambda-2\rho_1)$$

and set

$$V(\Lambda)=U\,T_+\otimes V_0(\Lambda-2\rho_1). \tag{15}$$

In view of the above remarks we have theorem 2.

Theorem 2. For $\Lambda\in D^+$, the module $V(\Lambda)$ is irreducible with highest weight Λ.

The module construction of (15) implicitly contains all information on the structure of irreducible L-modules. In particular it enables, in principle, a systematic determination of the structure of all irreducible atypical L-modules for a type-I basic classical Lie superalgebra.

3. Specific examples

With the induced module construction of (15), the L-module $V(\Lambda)$ appears as the unique irreducible submodule of the induced module $\bar{V}_{(-)}(\Lambda-2\rho_1)$. In this section we illustrate the utility of this construction with the examples of the identity and vector representations of $\mathrm{gl}(m|n)$ and all irreducible representations of $\mathrm{gl}(2|1)$. (We note that our previous results obtained for $A(m,n)$ extend to $\mathrm{gl}(m|n)$ with trivial modifications.)

Throughout we adopt the convenient index notation

$$\dot{\mu}=m+\mu \qquad 1\leqslant\mu\leqslant n.$$

With this convention the generators E_{ab} $(1\leqslant a,b\leqslant n+m)$ of $\mathrm{L}=\mathrm{gl}(m|n)$ are given by the $\mathrm{L}_0=\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ generators:

$$E_{ij}(1\leqslant i,j\leqslant m) \qquad E_{\dot{\mu}\dot{\nu}}(1\leqslant\mu,\nu\leqslant n)$$

together with the odd generators $E_{i\dot{\mu}}$, $E_{\dot{\mu}i}$ (spanning the odd space L_1) which transform, under commutation with the $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ generators, as the representations $(1,\dot{0})\otimes(\dot{0},-1)$, $(\dot{0},-1)\otimes(1,\dot{0})$, respectively. The Lie superalgebra structure is then completed by the graded commutation relations

$$[E_{i\dot{\mu}},E_{\dot{\nu}j}]=\delta_{\mu\nu}E_{ij}+\delta_{ij}E_{\dot{\nu}\dot{\mu}}$$
$$[E_{i\dot{\mu}},E_{j\dot{\nu}}]=[E_{\dot{\mu}i},E_{\dot{\nu}j}]=0.$$

Throughout we denote the highest weights of irreducible $\mathrm{gl}(m|n)$ modules by $\Lambda=(\lambda\,|\,\mu)$ where λ, μ denote highest weights of irreducible $\mathrm{gl}(m)$ and $\mathrm{gl}(n)$ modules, respectively.

For the case at hand the operator T_+ of (4) may be written

$$T_+=E_{1\dot{1}}E_{2\dot{1}}\dots E_{m\dot{1}}E_{1\dot{2}}\dots E_{m\dot{2}}\dots E_{m\dot{n}}$$

and transforms as the one-dimensional irreducible representation of $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ with highest weight $2\rho_1$, viz

$$\begin{aligned}&[E_{ij},T_+]=n\delta_{ij}T_+\\&[E_{\dot{\mu}\dot{\nu}},T_+]=-m\delta_{\mu\nu}T_+.\end{aligned} \tag{16}$$

We also find it convenient to introduce the operators $\bar{\Psi}_i^{\mu}$ defined by

$$E_{j\dot{\nu}}\bar{\Psi}_i^{\mu}=\delta_\nu^\mu\delta_{ji}T_+ \tag{17}$$

which transform as the irreducible representation of $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ with highest weight $\Lambda=(\dot{0},-1\,|\,1,\dot{0})+2\rho_1$, i.e.

$$\begin{aligned}&[E_{ij},\bar{\Psi}_k^{\mu}]=-\delta_{ik}\bar{\Psi}_j^{\mu}+n\delta_{ij}\bar{\Psi}_k^{\mu}\\&[E_{\dot{\mu}\dot{\nu}},\bar{\Psi}_i^{\tau}]=\delta_\nu^\tau\bar{\Psi}_i^{\mu}-m\delta_{\mu\nu}\bar{\Psi}_i^{\tau}.\end{aligned} \tag{18}$$

We similarly introduce the tensors

$$\bar{\Psi}_{ij}^{\mu\nu}=-\bar{\Psi}_{ji}^{\nu\mu}$$

defined by

$$E_{i\dot{\mu}}\bar{\Psi}_{jk}^{\nu\tau}=\delta_\mu^\nu\delta_{ij}\bar{\Psi}_k^{\tau}-\delta_\mu^\tau\delta_{ik}\bar{\Psi}_j^{\nu} \tag{19}$$

which transform according to

$$[E_{ij},\bar{\Psi}_{kl}^{\mu\nu}]=n\delta_{ij}\bar{\Psi}_{kl}^{\mu\nu}-\delta_{ik}\bar{\Psi}_{jl}^{\mu\nu}-\delta_{il}\bar{\Psi}_{kj}^{\mu\nu}$$
$$[E_{\dot{\mu}\dot{\nu}},\bar{\Psi}_{ij}^{\tau\sigma}]=\delta_\nu^\tau\bar{\Psi}_{ij}^{\mu\sigma}+\delta_\nu^\sigma\bar{\Psi}_{ij}^{\tau\mu}-m\delta_{\mu\nu}\bar{\Psi}_{ij}^{\tau\sigma}.$$

In a similar way higher-order tensors may be introduced, although this will not be necessary for the present treatment.

It is easily verified, in view of (16-19), that the following graded commutation relations hold (see the appendix):

$$\begin{aligned} &[E_{\dot\mu i}, T_+] = \bar\Psi_j^{\mu}(E-m)_{ji} + \bar\Psi_i^{\nu}(E+n)_{\dot\mu\dot\nu} \\ &[E_{\dot\mu i}, \bar\Psi_j^{\nu}] = \bar\Psi_{kj}^{\mu\nu}(E-m)_{ki} + \bar\Psi_{ij}^{\tau\nu}(E+n)_{\dot\mu\dot\tau} + \bar\Psi_{ij}^{\nu\mu} \end{aligned} \tag{20}$$

(summation over repeated indices assumed) where $(E-m)_{ij}$ is shorthand notation for $E_{ij} - m\delta_{ij}$, etc. The tensors defined by (17) and (19) and the relations of (20) play a fundamental role in the construction of irreducible $\mathrm{gl}(m\,|\,n)$ modules via equation (15). We conclude this section with some illustrative examples.

Identity representation. Let e_0 be the basis vector of the one-dimensional representation of $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ with highest weight

$$-2\rho_1 = (-\dot n\,|\,\dot m)$$

and, following the induced module construction of (15), define

$$E_{\dot\mu i} e_0 = 0.$$

We thus consider the irreducible $\mathrm{gl}(m\,|\,n)$ module generated by the highest-weight vector

$$\Omega_0 = T_+ \otimes e_0 \tag{21}$$

of trivial weight $(\dot 0\,|\,\dot 0)$. To determine the action of the $\mathrm{gl}(m\,|\,n)$ generators on the state (21) we have, in accordance with (16),

$$E_{ij}\Omega_0 = E_{\dot\mu\dot\nu}\Omega_0 = E_{i\dot\mu}\Omega_0 = 0.$$

For the remaining generators $E_{\dot\mu i}$ we may employ (20), according to which we obtain

$$\begin{aligned} E_{\dot\mu i}\Omega_0 &= [E_{\dot\mu i}, T_+]e_0 \\ &= \bar\Psi_j^{\mu}\otimes(E-m)_{ji}e_0 + \bar\Psi_i^{\nu}\otimes(E+n)_{\dot\mu\dot\nu}e_0 \\ &= -(m+n)\bar\Psi_i^{\mu}\otimes e_0 + (m+n)\bar\Psi_i^{\mu}\otimes e_0 = 0 \end{aligned}$$

where we have applied the results

$$E_{ij}e_0 = -n\delta_{ij}e_0 \qquad E_{\dot\mu\dot\nu}e_0 = m\delta_{\mu\nu}e_0.$$

The state (21) therefore gives rise to the trivial one-dimensional representation of $\mathrm{gl}(m\,|\,n)$ as required. This module is the unique irreducible $\mathrm{gl}(m\,|\,n)$ module occuring in the lowest-weight Kac-module

$$\mathrm{U}_+ \otimes_{\bar{\mathrm{U}}_-} e_0.$$

Vector representation. Let $\bar e^i$ $(i = 1, \ldots, m)$ constitute the basis vectors of the irreducible representation of $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ with highest-weight $\delta = (1, \dot 0\,|\,\dot 0) - 2\rho_1$, viz

$$\begin{aligned} E_{ij}\bar e^k &= \delta_j^k \bar e^i - n\delta_{ij}\bar e^k \\ E_{\dot\mu\dot\nu}\bar e^k &= m\delta_{\mu\nu}\bar e^k. \end{aligned} \tag{22}$$

Following the induced module construction of (15), we introduce the vectors

$$\Omega^k = T_+ \otimes \bar e^k \qquad E_{\dot\mu i}\bar e^k = 0$$

which generate an irreducible $\mathrm{gl}(m\,|\,n)$ module with highest weight $(1\dot 0\,|\,\dot 0)$.

We have, in accordance with (20), (22),

$$\begin{aligned} E_{\mu i}\Omega^k &= [E_{\mu i}, T_+]\bar{e}^k \\ &= \bar{\Psi}^\mu_j \otimes (E-m)_{ji}\bar{e}^k + \bar{\Psi}^\nu_i \otimes (E+n)_{\mu\nu}\bar{e}^k \\ &= \delta^k_i \Omega^\mu \end{aligned}$$

where Ω^μ $(\mu = 1, \ldots, n)$ is defined by

$$\Omega^\mu = \bar{\Psi}^\mu_i \otimes \bar{e}^i.$$

It is easily seen that the $\mathrm{gl}(m)\oplus\mathrm{gl}(n)$ generators act on the states Ω^k, Ω^μ according to

$$E_{ij}\Omega^k = \delta^k_j \Omega^i \qquad E_{\mu\nu}\Omega^i = 0$$

$$E_{\mu\nu}\Omega^\tau = \delta^\tau_\nu \Omega^\mu \qquad E_{ij}\Omega^\mu = 0.$$

Also, in view of (17), we have

$$E_{i\mu}\Omega^\nu = \delta^\nu_\mu \Omega^i$$

and, finally, from (20) we obtain

$$\begin{aligned} E_{\nu j}\Omega^\mu &= [E_{\nu j}, \bar{\Psi}^\mu_i]\bar{e}^i \\ &= \bar{\Psi}^{\nu\mu}_{li} \otimes (E-m)_{lj}\bar{e}^i + \bar{\Psi}^{\tau\mu}_{ji} \otimes (E+n)_{\nu\tau}\bar{e}^i + \bar{\Psi}^{\mu\nu}_{ji} \otimes \bar{e}^i \\ &= (\bar{\Psi}^{\nu\mu}_{ij} + \bar{\Psi}^{\mu\nu}_{ji}) \otimes \bar{e}^i = 0 \end{aligned}$$

where we have applied (22).

It follows that the states Ω^i, Ω^μ span the irreducible $(n+m)$-dimensional $\mathrm{gl}(m|n)$ module corresponding to the vector representation as required. In this induced module approach, the above irreducible $\mathrm{gl}(m|n)$ module appears as the unique irreducible submodule of the induced module

$$\mathrm{U}_+ \otimes_{\bar{\mathrm{U}}_-} V_0(\delta).$$

Irreducible representations of gl(2/1). In the case of $\mathrm{gl}(2|1)$, we denote our $\mathrm{gl}(2)\oplus\mathrm{gl}(1)$ generators by E^i_j $(1 \leqslant i, j \leqslant 2)$, Ω respectively, and denote the corresponding odd generators by Ψ^i, Ψ_i $(i = 1, 2)$, which satisfy the graded commutation relations

$$[E^i_j, \Psi^k] = \delta^k_j \Psi^i \qquad [E^i_j, \Psi_k] = -\delta^i_k \Psi_j$$

$$[\Omega, \Psi^k] = -\Psi^k \qquad [\Omega, \Psi_k] = \Psi_k$$

$$[\Psi^i, \Psi_j] = \delta^i_j \Omega + E^i_j.$$

In this case we have two odd positive roots:

$$\alpha_1 = (1, 0|-1) \qquad \alpha_2 = (0, 1|-1)$$

so that

$$2\rho_1 = (\alpha_1 + \alpha_2) = (1, 1|-2)$$

and

$$\rho = \rho_0 - \rho_1 = (0, -1|1).$$

The tensors of (17) and (19) in the case of gl(2|1) may be simply expressed (omitting the superfluous superscripts μ, ν, etc)

$$T_+ = \Psi^1\Psi^2 \qquad \bar{\Psi}_1 = \Psi^2 \qquad \bar{\Psi}_2 = -\Psi^1$$

$$\bar{\Psi}_{21} = -\bar{\Psi}_{12} = 1 \qquad \bar{\Psi}_{11} = \bar{\Psi}_{22} = 0$$

and the graded commutation relations of (20) reduce to

$$\begin{aligned} [\Psi_i, T_+] &= \bar{\Psi}_j(E+\Omega-1)^j_i \\ [\Psi_i, \bar{\Psi}_j] &= \bar{\Psi}_{kj}(E+\Omega)^k_i. \end{aligned} \tag{23}$$

Following the construction of (15), to construct the irreducible gl(2|1) module with highest weight $\Lambda = (\lambda_1, \lambda_2|\omega)$ we introduce the gl(2)⊕gl(1) module $V_0(\Lambda - 2\rho_1)$ and define

$$L_- V_0(\Lambda - 2\rho_1) = 0.$$

The states

$$\bar{v} = T_+ \otimes v \qquad v \in V_0(\Lambda - 2\rho_1)$$

then generate the unique irreducible gl(2|1) submodule of the induced module

$$U_+ \otimes_{\bar{U}_-} V_0(\Lambda - 2\rho_1)$$

with highest weight Λ.

We have, in view of (23),

$$\Psi_i \bar{v} = \bar{\Psi}_j \otimes (E+\Omega-1)^j_i v \tag{24}$$

where $\bar{v}$ belongs to the irreducible representation of gl(2)⊕gl(1) with highest weight Λ. Hence the state $\Psi_i\bar{v}$ transforms as a state in the tensor product module

$$V_0(\dot{0}, -1|1) \otimes V_0(\Lambda) = V_0(\Lambda - \alpha_1) \oplus V_0(\Lambda - \alpha_2).$$

To obtain the correct gl(2)⊕gl(1) symmetry adapted states, we employ the shift component formalism of Green (1971), according to which the operators Ψ_i may be resolved into shift components according to

$$\Psi_i = \Psi[1]_i + \Psi[2]_i \qquad \Psi[r]_i = \Psi_j P[r]^j_i$$

where

$$P[1] = \left(\frac{E - \varepsilon_2}{\varepsilon_1 - \varepsilon_2}\right) \qquad P[2] = \left(\frac{E - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}\right)$$

and ε_1, ε_2 are gl(2)-invariants which take constant values

$$\varepsilon_1 = \lambda_1 + 1 \qquad \varepsilon_2 = \lambda_2$$

on an irreducible gl(2) module with highest weight $\lambda = (\lambda_1, \lambda_2)$. In view of the gl(2) characteristic identity (Green 1971) we have

$$E^i_j P[r]^j_k = \varepsilon_r P[r]^i_k.$$

Hence we have

$$\Psi[r]_i\bar{v} = \Psi_j P[r]^j_i \bar{v} \in V_0(\Lambda - \alpha_r) \qquad r = 1, 2$$

for $\bar{v} \in V_0(\Lambda)$. For each $r = 1, 2$, the above states either span the irreducible gl(2)⊕gl(1) module with highest weight $\Lambda - \alpha_r$ or else are zero for all i and $\bar{v} \in V_0(\Lambda)$. Now, in view of (24), we have

$$\begin{aligned}\Psi[r]_i\bar{v} &= \Psi_j T_+ \otimes P[r]^j_i v \\ &= \bar{\Psi}_k \otimes (E + \Omega - 1)^k_j P[r]^j_i v \\ &= (\Lambda + \rho, \alpha_r)\bar{\Psi}_k \otimes P[r]^k_i v\end{aligned}$$

where we have employed the gl(2) characteristic identity together with the easily established relations

$$(\varepsilon_r + \Omega - 1)v = (\Lambda + \rho, \alpha_r)v \qquad \text{for } v \in V_0(\Lambda - 2\rho_1).$$

Hence it follows that the irreducible gl(2)⊕gl(1) module $V_0(\Lambda - \alpha_r)$ occurs in $V(\Lambda)$ if and only if $\Lambda - \alpha_r$ is dominant and $(\Lambda + \rho, \alpha_r) \neq 0$ $(r = 1, 2)$.

Finally we have the states

$$\begin{aligned}\Psi_2\Psi_1\bar{v} &= \Psi_2[\Psi_1, T_+] \otimes v \\ &= \Psi_2\bar{\Psi}_k \otimes (E + \Omega - 1)^k_1 v \\ &= \bar{\Psi}_{ik} \otimes (E + \Omega)^i_2 (E + \Omega - 1)^k_1 v \\ &= 1 \otimes \Delta'_0 v\end{aligned}$$

where Δ'_0 is the gl(2)⊕gl(1) invariant

$$\Delta'_0 = (E + \Omega)^2_2 (E + \Omega - 1)^1_1 - E^1_2 E^2_1$$

and we have applied the result: $\bar{\Psi}_{21} = 1$, $\bar{\Psi}_{ij} = -\bar{\Psi}_{ji}$. Using the easily established result

$$\Delta'_0 v = (\Lambda + \rho, \alpha_1)(\Lambda + \rho, \alpha_2)v \qquad v \in V_0(\Lambda - 2\rho_1)$$

we thus obtain (cf (9), (11) and (14))

$$\Psi_2\Psi_1 T_+ \otimes v = (\Lambda + \rho_1, \alpha_1)(\Lambda + \rho_1, \alpha_2)(1 \otimes v) \in V_0(\Lambda - 2\rho_1).$$

Hence the representation $V_0(\Lambda - 2\rho_1)$ only occurs in $V(\Lambda)$ if $(\Lambda + \rho, \alpha_r) \neq 0$ for $r = 1, 2$, i.e. if $V(\Lambda)$ is typical.

Thus, if $V(\Lambda)$ is typical we have a gl(2)⊕gl(1) module decomposition

$$V(\Lambda) = V_0(\Lambda) \oplus V_0(\Lambda - \alpha_1) \oplus V_0(\Lambda - \alpha_2) \oplus V_0(\Lambda - 2\rho_1)$$

whilst if $V(\Lambda)$ is atypical we have a decomposition

$$V(\Lambda) = V_0(\Lambda) \oplus V_0(\Lambda - \alpha)$$

where α is the unique odd positive root such that $(\Lambda + \rho, \alpha) \neq 0$. (Note that, for gl(2|1), only singly atypical representations can occur.) It should be remarked that, in the above decompositions, it is implicitly assumed that all modules corresponding to non-dominant highest weights are trivially zero.

The methods outlined above are extendable, in principle, to all irreducible representations of gl$(m|n)$, as will be demonstrated for the case $n = 1$ in a forthcoming publication.

4. Conclusion

We have explicitly constructed all finite-dimensional irreducible typical and atypical modules for type-I basic classical Lie superalgebras. In the typical case our results are simply an elaboration of those obtained previously by Kac (1978), but with greater emphasis on the role of lemma 1. This result in fact appears implicitly in Kac (1978, proposition (2.9)), but its use for the construction of atypical modules is not noted or exploited. However, the construction of atypical modules, via theorem 2, is potentially very useful, as illustrated in § 3, and in particular may be applied to the direct construction of representation matrices, at least for low-lying atypical irreducible representations, obtained previously by other methods (Thierry-Mieg 1983). Moreover this construction is useful for the determination of branching rules (and hence character formulae) for atypical modules. This method will be illustrated in a forthcoming publication where the $gl(n|1)\downarrow gl(n)$ branching rules will be given for all typical and atypical irreducible representations of $gl(n|1)$.

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Appendix

From the definition of the $\bar{\Psi}_i^{\mu}$ we clearly have

$$[E_{\dot{\mu}i}, T_+] = \sum_{\nu,j} \bar{\Psi}_j^{\nu} \otimes C_{\nu}^{j} \tag{A1}$$

for suitable operators C_{ν}^{j} in the enveloping algebra of $L_0 = gl(m) \oplus gl(n)$. Applying $E_{k\dot{\tau}}$ to the left of this equation we obtain, using (17),

$$\begin{aligned} T_+ \otimes C_{\tau}^{k} &= E_{k\dot{\tau}} \sum_{\nu,j} \bar{\Psi}_j^{\nu} \otimes C_{\nu}^{j} \\ &= E_{k\dot{\tau}}[E_{\dot{\mu}i}, T_+] \\ &= [E_{\dot{\mu}i}, E_{k\dot{\tau}}]T_+ - [E_{\dot{\mu}i}, E_{k\dot{\tau}}T_+] \\ &= (\delta_{ki}E_{\dot{\mu}\dot{\tau}} + \delta_{\mu\tau}E_{ki})T_+ \\ &= T_+ \otimes (n\delta_{ki}\delta_{\mu\tau} + E_{ki}\delta_{\mu\tau} - m\delta_{\mu\tau}\delta_{ki} + \delta_{ki}E_{\dot{\mu}\dot{\tau}}) \end{aligned}$$

or

$$C_{\tau}^{k} = \delta_{\mu\tau}(E+n)_{ki} + \delta_{ki}(E-m)_{\dot{\mu}\dot{\tau}}.$$

Substituting into (A1) we obtain

$$[E_{\dot{\mu}i}, T_+] = \bar{\Psi}_j^{\mu}(E-m)_{ji} + \bar{\Psi}_i^{\nu}(E+n)_{\dot{\mu}\dot{\nu}}.$$

In a similar way, using (19), we obtain

$$[E_{\dot{\mu}i}, \bar{\Psi}_j^{\nu}] = \bar{\Psi}_{kj}^{\mu\nu}(E-m)_{ki} + \bar{\Psi}_{ij}^{\tau\nu}(E+n)_{\dot{\mu}\dot{\tau}} + \bar{\Psi}_{ij}^{\nu\mu}$$

which is (20) as required. A generalisation of these relations, for higher tank tensors of $gl(n|1)$, will be given in a forthcoming publication.

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